



LISBOA SCHOOL OF ECONOMICS & MANAGEMENT

MASTERS IN ACTUARIAL SCIENCE

Risk Models

20/01/2020

1st part of the exam

Time allowed: 2 hours

Instructions:

1. This paper contains 6 questions and comprises 3 pages including the title page.
2. Enter all requested details on the cover sheet.
3. You have 10 minutes reading time. You must not start writing your answers until instructed to do so.
4. Number the pages of the paper where you are going to write your answers.
5. Attempt all questions.
6. Begin your answer to each of the questions on a new page.
7. Marks are shown in brackets. Total marks: 140.
8. Show calculations where appropriate.
9. An approved calculator may be used.
10. The distributed formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed formulary.

1. **[10]** An investigation about the performance and timing of corporate acquisitions discovered that, in a random sample of 2778 firms, 748 announced one or more acquisitions during the last year. Does the sample provide enough evidence to indicate that the true percentage of all firms that announced one or more acquisitions during the last year is less than 30%? Present the p-value of the test and conclude.

2. You are given the following (sorted) sample of 10 payments:

4 4 5 5 5 6 7 7 7 9

a) **[10]** Estimate $f(5.5)$ using the following kernel $k_y(x) = 0.75(1-x+y)(1+x-y)$, $y-1 < x < y+1$.

b) **[5]** Present the conditions that should be fulfilled by a function $k_y(x)$ to be used as a kernel function.

3. A random sample of 50 claim amounts is (after being sorted) the following

10.7	12.0	23.2	29.5	30.3	31.5	36.7	36.9	38.5	45.5
49.3	50.4	51.4	64.7	67.8	68.9	71.3	72.1	75.0	75.2
77.6	78.2	81.1	83.4	87.1	89.4	89.9	91.3	92.0	94.0
96.1	96.9	100.7	101.4	106.6	106.6	107.2	107.3	124.3	127.5
142.1	143.3	147.0	158.5	161.4	161.5	163.9	179.8	188.0	236.7

You can use the fact that $\sum_{i=1}^{50} x_i = 4561.7$ and $\sum_{i=1}^{50} x_i^2 = 534148.6$.

a) **[15]** Using a non-parametric framework present an **estimator** for $S(90)$. Is this estimator consistent?

b) **[10]** Using the estimator presented in the previous question, compute a 90% confidence interval for $S(90)$.

c) **[10]** Now assume that we know that claim amounts follow a Weibull distribution. Using the percentile matching method (percentiles 20% and 80%) obtain a new estimate for $S(90)$.

d) **[15]** Assuming that additionally, we knew that $\tau = 2$ obtain a maximum likelihood estimate for θ . Also estimate the variance of your estimator.

e) **[10]** Using the asymptotic distribution of the maximum likelihood estimators and the delta method, compute a 90% confidence interval for $S(90)$ assuming the Weibull distribution with $\tau = 2$

4. From a Bayesian framework assume that our model is given by $f(x|\theta) = \theta^2 x e^{-\theta x}$, $x > 0$ and $\theta > 0$, and that the prior for θ is given by a Gamma with parameters 0.5 and 0.1, i.e. $\pi(\theta) = \frac{\theta^{0.5-1} e^{-\theta/0.1}}{\Gamma(0.5) 0.1^{0.5}}$, $\theta > 0$. Also assume that we observed the random sample presented in exercise 3.
- [10]** Obtain the posterior (if you are unable to solve this question, you can assume that the posterior is a gamma distribution with parameters 5.05 and 0.0001 – which is not the correct answer – to answer the remaining questions)
 - [10]** Compute the predictive distribution.
 - [10]** Using Bayes Central Limit Theorem compute a 95% HPD interval for θ . Is the use of Bayes Central Limit Theorem the more adequate procedure? Explain.
5. **[15]** You are given the following sample (30 60 70 100 160). Test ($\alpha = 0.05$), using the Kolmogorov-Smirnov procedure, if it is acceptable to consider that the population follows a Weibull distribution with $\theta = 100$ and $\tau = 2$, i.e. $F(x|\theta, \tau) = 1 - e^{-(x/\theta)^\tau}$.
6. **[10]** You are given the following observations (“+” means a right censored value) from a Pareto population with $\theta = 20$
- 2 4 4+ 5 6+ 8
- You also know that observation 1 and observation 3 are truncated at 1 and 3 respectively. Find the maximum likelihood estimate of α

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Solutions

1.

$$H_0: \theta \geq 0.3 \quad H_1: \theta < 0.3$$

$$\text{Test Statistic (large samples, CLT): } Z = \frac{\bar{X} - 0.3}{\sqrt{0.3 \times 0.7 / 2778}} \stackrel{a}{\sim} N(0;1)$$

$$\text{p-value} = P(Z \leq z_{obs}) = P\left(Z \leq \frac{748 / 2778 - 0.3}{\sqrt{0.21 / 2778}}\right) = P(Z \leq -3.53575) = 1 - 0.99980 = 0.0002$$

So we can conclude that the true percentage of all firms that announced one or more acquisitions during the last year is less than 30%

2.

a.

j	1	2	3	4	5
y_j	4	5	6	7	9
$p(y_j)$	0.2	0.3	0.1	0.3	0.1
$k_{y_j}(5.5)$	0	$0.75 \times 0.5 \times 1.5$	$0.75 \times 1.5 \times 0.5$	0	0

$$\text{Then } \hat{f}(5.5) = 0 + 0.3 \times (0.75 \times 0.5 \times 1.5) + 0.1 \times (0.75 \times 1.5 \times 0.5) + 0 + 0 = 0.225$$

b.

To be used as a kernel 2 conditions are mandatory, $k_y(x) \geq 0, \forall x$ and $\int_{-\infty}^{+\infty} k_y(x) dx = 1$, and optionally we can add a third condition to keep the expected value unchanged,

$$\int_{-\infty}^{+\infty} x k_y(x) dx = y$$

3.

a) The estimator is given by $S_n(90) = \frac{\#\{X_i > 90\}}{n}$.

As we know, $\#\{X_i > 90\} \sim B(n, S(90))$ and then

$$E(S_n(90)) = E\left(\frac{\#\{X_i > 90\}}{n}\right) = \frac{E(\#\{X_i > 90\})}{n} = \frac{nS(90)}{n} = S(90)$$

$$\text{var}(S_n(90)) = \text{var}\left(\frac{\#\{X_i > 90\}}{n}\right) = \frac{\text{var}(\#\{X_i > 90\})}{n^2} = \frac{nS(90)(1-S(90))}{n^2} = \frac{S(90)(1-S(90))}{n}$$

As the estimator is unbiased $E(S_n(90)) = S(90)$, it will be enough to show that

$\lim_{n \rightarrow \infty} \text{var}(S_n(90)) = 0$ to prove that it is consistent.

As $\lim_{n \rightarrow \infty} \text{var}(S_n(90)) = \lim_{n \rightarrow \infty} \frac{S(90)(1-S(90))}{n} = 0$, the proof is done.

b)

The estimate of $S(90)$ is $S_n(90) = \frac{\#\{x_i > 90\}}{n} = \frac{23}{50} = 0.46$. The estimate of the variance of the corresponding estimator is

$\hat{\text{var}}(S_n(90)) = \frac{S_n(90)(1-S_n(90))}{n} = \frac{0.46 \times 0.54}{50} = \frac{0.2484}{50} = 0.004968$ and the 90% CI is given by

$S_n(90) \pm 1.645 \sqrt{S_n(90) \times (1-S_n(90)) / n}$, i.e. (0.3441; 0.5759)

c)

Empirical quantiles:

$$(n+1) \times 0.2 = 51 \times 0.2 = 10.2 \quad \tilde{\pi}_{0.2} = 0.8x_{(10)} + 0.2x_{(11)} = 0.8 \times 45.5 + 0.2 \times 49.3 = 46.26$$

$$(n+1) \times 0.8 = 51 \times 0.8 = 40.8 \quad \tilde{\pi}_{0.8} = 0.2x_{(40)} + 0.8x_{(41)} = 0.2 \times 127.5 + 0.8 \times 142.1 = 139.18$$

Now we must solve the system
$$\begin{cases} 0.2 = 1 - e^{-(\tilde{\pi}_{0.2}/\theta)^\tau} \\ 0.8 = 1 - e^{-(\tilde{\pi}_{0.8}/\theta)^\tau} \end{cases}$$

$$\begin{cases} 0.2 = 1 - e^{-(\tilde{\pi}_{0.2}/\theta)^\tau} \\ 0.8 = 1 - e^{-(\tilde{\pi}_{0.8}/\theta)^\tau} \end{cases} \Leftrightarrow \begin{cases} e^{-(\tilde{\pi}_{0.2}/\theta)^\tau} = 0.8 \\ e^{-(\tilde{\pi}_{0.8}/\theta)^\tau} = 0.2 \end{cases} \Leftrightarrow \begin{cases} (\tilde{\pi}_{0.2}/\theta)^\tau = -\ln(0.8) \\ (\tilde{\pi}_{0.8}/\theta)^\tau = -\ln(0.2) \end{cases} \Leftrightarrow \begin{cases} (\tilde{\pi}_{0.2}/\theta)^\tau = -\ln(0.8) \\ (\tilde{\pi}_{0.8}/\tilde{\pi}_{0.2})^\tau = (\ln(0.2)/\ln(0.8)) \end{cases}$$

Now, solving the second equation we get

$$\tau \ln(\tilde{\pi}_{0.8}/\tilde{\pi}_{0.2}) = \ln(\ln(0.2)/\ln(0.8)) \Leftrightarrow \tau = \frac{\ln(\ln(0.2)/\ln(0.8))}{\ln(\tilde{\pi}_{0.8}/\tilde{\pi}_{0.2})} = 1.793774$$

And then

$$\frac{\tilde{\pi}_{0.2}}{\theta} = (-\ln(0.8))^{1/\tau} \Leftrightarrow \theta = \tilde{\pi}_{0.2} (-\ln(0.8))^{-(1/\tau)} \Leftrightarrow \theta = 106.7479$$

The new estimate for $S(90)$ is $\tilde{S}(90) = e^{-(90/106.7479)^{1.793774}} = 0.4788854$

d)

$$f(x|\theta) = 2x\theta^{-2}e^{-(x/\theta)^2}$$

$$\ell(\theta) = \sum_{i=1}^n \ln f(x_i|\theta) = \sum_{i=1}^n \left(\ln 2 + \ln x_i - 2\ln \theta - \left(\frac{x_i}{\theta}\right)^2 \right)$$

$$\ell'(\theta) = \sum_{i=1}^n (-2\theta^{-1} + 2x_i^2\theta^{-3}) = -2n\theta^{-1} + 2\theta^{-3} \sum_{i=1}^n x_i^2$$

$$\ell'(\theta) = 0 \Leftrightarrow 2n\theta^{-1} = 2\theta^{-3} \sum_{i=1}^n x_i^2 \Leftrightarrow \theta^2 = \frac{\sum_{i=1}^n x_i^2}{n} = \frac{534148.6}{50} = 10682.97$$

$$\ell''(\theta) = 2n\theta^{-2} - 6\theta^{-4} \sum_{i=1}^n x_i^2 \quad \text{and} \quad \ell''(\theta)|_{\theta: \ell'(\theta)=0} = 2n \frac{n}{\sum_{i=1}^n x_i^2} - 6 \frac{n^2}{\left(\sum_{i=1}^n x_i^2\right)^2} \sum_{i=1}^n x_i^2 = -\frac{4n^2}{\sum_{i=1}^n x_i^2} < 0 .$$

$$\text{Then } \hat{\theta} = \sqrt{\frac{\sum_{i=1}^n x_i^2}{n}} = 103.3585$$

$$\text{var}(\hat{\theta}) = \left(-\ell''(\hat{\theta})\right)^{-1} = \left(\frac{4n^2}{\sum_{i=1}^n x_i^2}\right)^{-1} = \frac{\sum_{i=1}^n x_i^2}{4n^2} = \frac{\hat{\theta}^2}{4n} = 53.41486$$

e)

$$S(90) = e^{-(90/\theta)^2} \quad \hat{S}(90) = e^{-(90/\hat{\theta})^2} = 0.4685$$

$$g(\theta) = e^{-(90/\theta)^2} \quad g'(\theta) = 2 \times 90^2 \theta^{-3} e^{-(90/\theta)^2} \quad g'(\hat{\theta}) = 0.006874$$

$$\text{var}(\hat{S}(90)) = \left(g'(\hat{\theta})\right)^2 \text{var}(\hat{\theta}) = \left(2 \times 90^2 \hat{\theta}^{-3} e^{-(90/\hat{\theta})^2}\right)^2 \frac{\hat{\theta}^2}{4n} = \frac{4 \times 90^4}{4n} \hat{\theta}^{-4} e^{-2(90/\hat{\theta})^2} = 0.0025237$$

Then the 90% CI is given by $0.4685 \pm 1.645 \sqrt{0.0025237}$ i.e. (0.3859; 0.5511)

4.

a.

$$f(x|\theta) = \theta^2 x e^{-\theta x}, \quad x > 0, \quad \theta > 0$$

$$\theta \sim G(a, b), \quad \text{i.e. } \pi(\theta) = \frac{\theta^{a-1} e^{-\theta/b}}{\Gamma(a)b^a} \propto \theta^{a-1} e^{-\theta/b}, \quad \text{with } a = 0.5 \text{ and } b = 0.1$$

$$L(\theta | x_1, x_2, \dots, x_n) = \prod_{i=1}^n (\theta^2 x_i e^{-\theta x_i}) \propto \theta^{2n} e^{-\theta t} \quad \text{with } t = \sum_{i=1}^n x_i = 4561.7$$

$$\pi_{\theta|x}(\theta) \propto \theta^{a-1} e^{-\theta/b} \theta^{2n} e^{-\theta t} = \theta^{2n+a-1} e^{-\theta(t+1/b)}, \quad \text{core of a Gamma distribution with parameters}$$

$2n+a$ and $(t+1/b)^{-1}$, i.e. 100.5 and 1/4571.7. The posterior is then

$$\pi_{\theta|x}(\theta) = \frac{\theta^{100.5-1} e^{-4571.7\theta}}{\Gamma(100.5)(4571.7)^{-100.5}}, \quad \theta > 0$$

b.

Predictive distribution

Let the posterior be a gamma with parameters a and b ($a = 100.5$ and $b = 1/4571.7$ for the correct solution or $a = 5.05$ and $b = 0.0001$ for the alternative hypothesis)

$$f(y) = \int_0^\infty \theta^2 y e^{-\theta y} \frac{\theta^{a-1} e^{-\theta/b}}{\Gamma(a) b^a} d\theta = \frac{y}{\Gamma(a) b^a} \int_0^\infty \theta^{2+a-1} e^{-\theta(y+1/b)} d\theta$$

$$= \frac{y \Gamma(2+a) (y+1/b)^{-(2+a)}}{\Gamma(a) b^a} = \frac{a(a+1)}{b^a} \frac{y}{(y+1/b)^{2+a}} \quad y > 0$$

Using $a = 100.5$ and $b = 1/4571.7$ (correct solution) we get

$$f(y) = \frac{100.5 \times 101.5 \times 4571.7^{100.5} \times y}{(y+4571.7)^{102.5}}, \quad y > 0$$

Using $a = 5.05$ and $b = 0.0001$ (alternative) we get $f(y) = \frac{5.05 \times 6.05 \times 10000^{5.05} \times y}{(y+10000)^{5.05}}, \quad y > 0$

c.

As $\theta | \mathbf{x} \sim G(100.5, 1/4571.7)$ then $E(\theta | \mathbf{x}) = \frac{100.5}{4571.7} = 0.02198$ and

$$\text{var}(\theta | \mathbf{x}) = \frac{100.5}{4571.7^2} = 4.8085 \times 10^{-6}$$

The 95% CI is then given by $0.02198 \pm 1.96 \sqrt{4.8085 \times 10^{-6}}$, (0.0177, 0.0263)

In this case, as the posterior is a gamma distribution, we can obtain an exact result using this distribution.

Using the alternative values, the expected value is 0.000505 and the variance is 5.05×10^{-8} and then the CI is $(6.454 \times 10^{-5}, 0.000945)$

5.

$H_0 : X \sim \text{Weibull}(\tau = 2, \theta = 100)$ $H_1 : \text{The null is false}$

$$F_0(x) = 1 - e^{-(x/100)^2}, \quad x \geq 0$$

i	x_i	$F_0(x_i)$	$(i-1)/n$	i/n	D^-	D^+
1	30	0.0861	0	0.2	0.0861	0.1139
2	60	0.3023	0.2	0.4	0.1023	0.0977
3	70	0.3874	0.4	0.6	0.0126	0.2126
4	100	0.6321	0.6	0.8	0.0321	0.1679
5	160	0.9227	0.8	1	0.1227	0.0773

$$D_n = 0.2126 \quad \text{Critical value } (\alpha = 0.05) = \frac{1.36}{\sqrt{5}} = 0.6082 \quad \text{Do not reject the null}$$

6.

$$\ell(\alpha) = \ln \frac{f(2|\alpha)}{S(1|\alpha)} + \ln f(4|\alpha) + \ln \frac{S(4|\alpha)}{S(3|\alpha)} + \ln f(5|\alpha) + \ln S(6|\alpha) + \ln f(8|\alpha)$$

$$\text{As } F(x) = 1 - \left(\frac{\theta}{x+\theta} \right)^\alpha, \text{ then } S(x) = \left(\frac{\theta}{x+\theta} \right)^\alpha \text{ and } \ln S(x) = \alpha \ln \theta - \alpha \ln(x+\theta)$$

$$f(x) = \frac{\alpha \theta^\alpha}{(x+\theta)^{\alpha+1}}, \text{ then } \ln f(x) = \ln \alpha + \alpha \ln \theta - (\alpha+1) \ln(x+\theta)$$

$$\begin{aligned} \ell(\alpha) &= \ln \alpha + \alpha \ln \theta - (\alpha+1) \ln(2+\theta) - (\alpha \ln \theta - \alpha \ln(1+\theta)) + \ln \alpha + \alpha \ln \theta - (\alpha+1) \ln(4+\theta) + \\ &\quad + \alpha \ln \theta - \alpha \ln(4+\theta) - (\alpha \ln \theta - \alpha \ln(3+\theta)) + \ln \alpha + \alpha \ln \theta - (\alpha+1) \ln(5+\theta) + \\ &\quad + \alpha \ln \theta - \alpha \ln(6+\theta) + \ln \alpha + \alpha \ln \theta - (\alpha+1) \ln(8+\theta) \\ &= 4 \ln \alpha + 4 \alpha \ln 20 - (\alpha+1) (\ln(22) + \ln(24) + \ln(25) + \ln(28)) + \alpha (\ln(21) - \ln(24) + \ln(23) - \ln(26)) \end{aligned}$$

$$\begin{aligned} \ell'(\alpha) &= \frac{4}{\alpha} + 4 \ln 20 - (\ln(22) + \ln(24) + \ln(25) + \ln(28)) + (\ln(21) - \ln(24) + \ln(23) - \ln(26)) \\ &= \frac{4}{\alpha} + 4 \ln 20 + \ln 21 - \ln 22 + \ln 23 - 2 \ln 24 - \ln 25 - \ln 26 - \ln 28 = \frac{4}{\alpha} - 1.09338 \end{aligned}$$

$$\ell'(\alpha) = 0 \Leftrightarrow \frac{4}{\alpha} = 1.09338 \Leftrightarrow \alpha = \frac{4}{1.09338} = 3.65838$$

$$\ell''(\alpha) = -\frac{4}{\alpha^2} < 0$$



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3. You have 10 minutes reading time. You must not start writing your answers until instructed to do so.
4. You are requested to summarize your answers on exam sheets. You can add any comments you think necessary to understand your answers.
5. At the end of the exam, you should submit your R files to Aquila using your usual username and password.
6. Attempt all questions.
7. Marks are shown in brackets. Total marks: 60.
8. The distributed formulary and the Formulae and Tables for Actuarial Examinations (the 2002 edition) may be used. Note that the parametrization used for the different distributions is that of the distributed formulary.

1. Atmospheric pollution can be related to many causes. File usair.csv (source gamlss.data library) presents six variables that can be related to atmospheric pollution for 41 US cities:

- Temp: Average annual temperature in °F
- Manuf: Number of manufacturing enterprises employing 20 or more workers
- Pop: Population size (1970 census) in thousands
- Wind: Average annual wind speed in miles per hour
- Precip: Average annual precipitation in inches
- Days: Average number of days with precipitation per year

Data is given in file usair.csv.

- a. **[5]** Assuming that the variable Temp is normally distributed with mean μ , test $H_0 : \mu \leq 50$ against $H_1 : \mu > 50$.
- b. **[15]** Using a PCA answer to the following questions:
 - i. Should we scale the variables? Explain.
 - ii. How many principal components should be retained? What is the proportion of the total variance that is explained by the retained components?
 - iii. Present the loadings for the retained components. Can you interpret them?

2. File Weibull.csv presents 100 observations from a population that follows a Weibull distribution with parameters τ and θ .

- a. **[15]** Obtain a maximum likelihood estimate for the unknown parameters and determine an asymptotic 95% confidence interval for θ .
- b. **[5]** Present a maximum likelihood estimate for $S(90)$.

3. **[20]** Assume that $X \sim G(100, 0.75)$ and that, given $X = x$, $Y \sim G(x/5, 10)$. Using simulation get an approximation to $E(Y)$ and to $P(Y > 200)$

Solution

1.

a. $H_0 : \mu \leq 50$ $H_1 : \mu > 50$.

Command: `t.test(Temp,alternative="greater",mu=50)`

One Sample t-test

data: Temp

t = 5.1059, df = 40, p-value = 4.222e-06

alternative hypothesis: true mean is greater than 50

95 percent confidence interval:

53.86272 Inf

sample estimates:

mean of x

55.76341

As the p-value is close to 0 we reject the null and conclude that the expected value of Temp is greater than 50 °F.

b. i. Yes we should scale the variables as they are measured using different scales.

ii.

Commands:

`x=cbind(Temp,Manuf,Pop,Wind,Precip,Days)`

`out.PCA=prcomp(x,scale=T,center=T)`

`out.PCA`

Standard deviations (1, .., p=6):

[1] 1.4819456 1.2247218 1.1809526 0.8719099 0.3384829 0.1855998

Rotation (n x k) = (6 x 6):

	PC1	PC2	PC3	PC4	PC5	PC6
Temp	-0.32964613	0.1275974	-0.67168611	-0.30645728	0.55805638	-0.13618780
Manuf	0.61154243	0.1680577	-0.27288633	0.13684076	-0.10204211	-0.70297051
Pop	0.57782195	0.2224533	-0.35037413	0.07248126	0.07806551	0.69464131
Wind	0.35383877	-0.1307915	0.29725334	-0.86942583	0.11326688	-0.02452501
Precip	-0.04080701	-0.6228578	-0.50456294	-0.17114826	-0.56818342	0.06062222
Days	0.23791593	-0.7077653	0.09308852	0.31130693	0.58000387	-0.02196062

Using Kaiser's criterion we retain 3 components (eigenvalues >1).

As the total variance is 6 (we have 6 scaled variables), the proportion retained is given by

$$\frac{1.4819456^2 + 1.2247218^2 + 1.1809526^2}{6} = 0.8485$$

Command: `sum(out.PCA$sdev[out.PCA$sdev>1]^2)/sum(out.PCA$sdev^2)`

[1] 0.8484592

iii.

Command: cor(x,out.PCA\$x[,1:3])

	PC1	PC2	PC3
Temp	-0.48851762	0.1562713	-0.7932295
Manuf	0.90627258	0.2058239	-0.3222658
Pop	0.85630067	0.2724433	-0.4137753
Wind	0.52436980	-0.1601832	0.3510421
Precip	-0.06047377	-0.7628275	-0.5958649
Days	0.35257846	-0.8668156	0.1099331

The first component is linked to Manuf and Pop that is to human impact

The second component is linked to Precip and Days, that is to precipitation.

The third component is linked to the Temperature.

2

a)

Commands:

```
dta=read.csv("F:/PCA dataSet/ML.weibull.csv",head=T)
minusloglik.weibull=function(param,x) {
  tau=param[1]; theta=param[2]
  return(-sum(dweibull(x,shape=tau,scale=theta,log=T)))
}
param.start=c(1,1)
out.ML=nlm(minusloglik.weibull,param.start,hessian=T,x=dta$x)
Warning messages:
1: In dweibull(x, shape = tau, scale = theta, log = T) : NaNs produced
2: In nlm(minusloglik.weibull, param.start, hessian = T, x = dta$x) :
  NA/Inf replaced by maximum positive value
3: In dweibull(x, shape = tau, scale = theta, log = T) : NaNs produced
4: In nlm(minusloglik.weibull, param.start, hessian = T, x = dta$x) :
  NA/Inf replaced by maximum positive value
5: In dweibull(x, shape = tau, scale = theta, log = T) : NaNs produced
6: In nlm(minusloglik.weibull, param.start, hessian = T, x = dta$x) :
  NA/Inf replaced by maximum positive value
7: In dweibull(x, shape = tau, scale = theta, log = T) : NaNs produced
8: In nlm(minusloglik.weibull, param.start, hessian = T, x = dta$x) :
  NA/Inf replaced by maximum positive value
out.ML
$minimum
[1] 488.0776
$estimate
[1] 3.398021 110.006524
$gradient
[1] -9.150428e-06 5.177612e-07
$hessian
      [,1]      [,2]
[1,] 15.7528115 -0.38905764
[2,] -0.3890576 0.09535387
$code
[1] 1
$iterations
[1] 25
```

The maximum likelihood estimates are $\hat{\tau} = 3.398021$ and $\hat{\theta} = 110.006524$
 To get a mle of the covariance matrix we must compute the inverse of the Hessian
 (remember that it is the hessian matrix for minus the loglikelihood). Then we will use
 element (2,2) of this matrix

Commands with output:

```
(theta.h=out.ML$estimate[2])
[1] 110.0065
(var.theta.h=solve(out.ML$hessian)[2,2])
[1] 11.66248
cbind(theta.h-1.96*sqrt(var.theta.h),theta.h+1.96*sqrt(var.theta.h))
      [,1] [,2]
[1,] 103.3131 116.7
```

Then the 95% asymptotic CI is (103.3131, 116.7)

b.

Command:

```
> (s90.h=pweibull(90,shape=out.ML$estimate[1],scale=out.ML$estimate[2],
lower.tail=F))
[1] 0.6031666
```

And then $\hat{S}(90) = 0.6031666$

3 Commands

```
NR=1000;
x=rgamma(NR,shape=100,scale=0.75)
y=rgamma(NR,shape=x/5,scale=10)
mean(y); mean(y>200)
[1] 149.2937
[1] 0.124
```

Then $E(Y) \approx 149.2937$ and $\hat{P}(Y > 200) \approx 0.124$

Note: An exact solution can be obtained for $E(Y)$ as

$$E(Y) = E(E(Y|X)) = E\left(\frac{10X}{5}\right) = 2E(X) = 2 \times 75 = 150$$

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